

ENERGY-MOMENTUM FOR ASYMPTOTICALLY ANTI-DE SITTER SPACETIMES

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ABSTRACT. We provide a definition of the total energy-momentum for asymptotically anti-de Sitter initial data sets which are asymptotic to t -slice in anti-de Sitter spacetime. The definition arises from the boundary terms in Witten's argument of the positive energy theorem. It reduces to Chruściel-Maerten-Tod's definition when $t = 0$. We prove the positive energy theorem for asymptotically anti-de Sitter spacetimes. We also verify that the total energy-momentum actually equals to Henneaux-Teitelboim's energy-momentum.

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1. INTRODUCTION

The positive energy theorem plays a fundamental role in general relativity. When the cosmological constant is zero and spacetimes are asymptotically flat, the positive energy theorem was first proved by Schoen and Yau [16, 17, 18], then by Witten [20, 15]. (See [2] for the definition of the ADM total energy-momentum.) We refer to [9, 4, 8, 22] for the case of high dimensional spacetimes.

When the cosmological constant is negative and spacetimes are asymptotically anti-de Sitter, initial data sets are asymptotically hyperbolic and the second fundamental forms are asymptotic to zero, and some special cases of the positive energy theorem were proved mathematically in [19, 6, 14, 7]. We refer to [1] for the extension of [2] in the negative cosmological constant and the physical proof of positivity, and to [3] for a definition for energy and other conserved quantities for asymptotically anti-de Sitter spacetimes in tetrad formalism. There is also another version of the positive energy theorem for asymptotically hyperbolic manifolds (e.g. [23, 21]), which represent initial data sets near null infinity in asymptotically flat spacetimes. In this case both the metrics and the second fundamental forms are asymptotic to the hyperbolic metric.

Recall that, with respect to the anti-de Sitter metric

$$\tilde{g} = -\frac{1+|x|^2}{1-|x|^2}dt^2 + \frac{4}{(1-|x|^2)^2} \sum_{i=1}^3 (dx^i)^2,$$

Chruściel, Maerten, and Tod [7] provided definitions of the total energy m_ν , $\nu = 0, 1, 2, 3$, the rest-frame angular momentum $j_{(i)}$ and the center of mass $c_{(i)}$, $i = 1, 2, 3$. By taking suitable coordinate transformations which ensure

$$m_{(1)} = m_{(2)} = m_{(3)} = c_{(3)} = j_{(2)} = j_{(3)} = 0, \quad (1.1)$$

they proved the energy inequality

$$m_{(0)} \geq \sqrt{|\vec{c}|^2 + |\vec{j}|^2 + 2|\vec{c} \times \vec{j}|} \quad (1.2)$$

where $\vec{m} = (m_{(1)}, m_{(2)}, m_{(3)})$, $\vec{c} = (c_{(1)}, c_{(2)}, c_{(3)})$ and $\vec{j} = (j_{(1)}, j_{(2)}, j_{(3)})$. However, it is not obvious that such coordinate transformations always exist. In this paper, we shall establish the general energy inequality without condition (1.1).

The anti-de Sitter spacetime is indeed the hyperboloid

$$\eta_{\alpha\beta} y^\alpha y^\beta = \frac{3}{\Lambda}, \quad \Lambda = -3\kappa^2 \ (\kappa > 0) \quad (1.3)$$

in $\mathbb{R}^{3,2}$ equipped with the metric

$$\eta_{\alpha\beta} dy^\alpha dy^\beta = -(dy^0)^2 + \sum_{i=1}^3 (dy^i)^2 - (dy^4)^2.$$

Under suitable choices of coordinates, the induced metric can be written as

$$\tilde{g} = -\cosh^2(\kappa r) dt^2 + dr^2 + \frac{\sinh^2(\kappa r)}{\kappa^2} (d\theta^2 + \sin^2 \theta d\psi^2). \quad (1.4)$$

The ten vectors

$$U_{\alpha\beta} = y_\alpha \frac{\partial}{\partial y^\beta} - y_\beta \frac{\partial}{\partial y^\alpha} \quad (1.5)$$

are Killing vectors generating rotations for $\mathbb{R}^{3,2}$. Note that $U_{\alpha\beta}$ depends on time t restricting on (1.3) equipped with the metric (1.4) (see Appendix).

We remark that the positive energy theorem for asymptotically de Sitter spacetimes is completely different, which is not always true [13, 12].

The paper is organized as follows. In Section 2, we provide a definition of the total energy-momentum for asymptotically anti-de Sitter initial data sets, which are asymptotic to t -slice in (1.4). This total energy-momentum arises from the boundary terms in Witten's argument of the positive energy theorem. And it reduces to the one given in [7] when $t = 0$. In Section 3, we prove the positive energy theorem for asymptotically anti-de Sitter spacetimes using Witten's argument. In Section 4, we verify the total energy-momentum equals to Henneaux-Teitelboim's energy-momentum [11]. In Appendix, we provide the restriction of $U_{\alpha\beta}$ on anti-de Sitter spacetime.

2. TOTAL ENERGY-MOMENTUM

In this section we will provide a definition of the total energy-momentum for asymptotically anti-de Sitter spacetimes. Let the coframe of (1.4) be

$$\check{e}^0 = \cosh(\kappa r)dt, \quad \check{e}^1 = dr, \quad \check{e}^2 = \frac{\sinh(\kappa r)}{\kappa}d\theta, \quad \check{e}^3 = \frac{\sinh(\kappa r)\sin\theta}{\kappa}d\psi$$

and denote \check{e}_α as its dual frame. For convenience, we fix the following Clifford representation throughout this paper although the total energy-momentum and the positivity do not depend on the specific representation.

$$\begin{aligned} \check{e}_0 &\mapsto \begin{pmatrix} & 1 & \\ 1 & & \\ & 1 & \end{pmatrix}, \quad \check{e}_1 \mapsto \begin{pmatrix} & -1 & \\ 1 & & \\ & -1 & \end{pmatrix}, \\ \check{e}_2 &\mapsto \begin{pmatrix} & & 1 \\ & 1 & \\ -1 & -1 & \end{pmatrix}, \quad \check{e}_3 \mapsto \sqrt{-1} \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & -1 & \end{pmatrix}. \end{aligned} \quad (2.1)$$

Then the imaginary Killing spinors Φ

$$\nabla_X^{AdS} \Phi + \frac{\kappa\sqrt{-1}}{2} X \cdot \Phi = 0 \quad \text{for each } X \text{ tangent to } N.$$

are of the form

$$\Phi_0 = \begin{pmatrix} u^+ e^{\frac{\kappa T}{2}} + u^- e^{-\frac{\kappa T}{2}} \\ v^+ e^{\frac{\kappa T}{2}} + v^- e^{-\frac{\kappa T}{2}} \\ -\sqrt{-1}u^+ e^{\frac{\kappa T}{2}} + \sqrt{-1}u^- e^{-\frac{\kappa T}{2}} \\ \sqrt{-1}v^+ e^{\frac{\kappa T}{2}} - \sqrt{-1}v^- e^{-\frac{\kappa T}{2}} \end{pmatrix} \quad (2.2)$$

where

$$\begin{aligned} u^+ &= \left(\lambda_1 \cos \frac{\kappa t}{2} + \lambda_3 \sin \frac{\kappa t}{2} \right) e^{\frac{\sqrt{-1}}{2}\psi} \sin \frac{\theta}{2} \\ &\quad + \left(\lambda_2 \cos \frac{\kappa t}{2} + \lambda_4 \sin \frac{\kappa t}{2} \right) e^{\frac{-\sqrt{-1}}{2}\psi} \cos \frac{\theta}{2}, \\ u^- &= \left(-\lambda_1 \sin \frac{\kappa t}{2} + \lambda_3 \cos \frac{\kappa t}{2} \right) e^{\frac{\sqrt{-1}}{2}\psi} \sin \frac{\theta}{2} \\ &\quad + \left(-\lambda_2 \sin \frac{\kappa t}{2} + \lambda_4 \cos \frac{\kappa t}{2} \right) e^{\frac{-\sqrt{-1}}{2}\psi} \cos \frac{\theta}{2}, \\ v^+ &= - \left(-\lambda_1 \sin \frac{\kappa t}{2} + \lambda_3 \cos \frac{\kappa t}{2} \right) e^{\frac{\sqrt{-1}}{2}\psi} \cos \frac{\theta}{2} \\ &\quad + \left(-\lambda_2 \sin \frac{\kappa t}{2} + \lambda_4 \cos \frac{\kappa t}{2} \right) e^{\frac{-\sqrt{-1}}{2}\psi} \sin \frac{\theta}{2}, \\ v^- &= - \left(\lambda_1 \cos \frac{\kappa t}{2} + \lambda_3 \sin \frac{\kappa t}{2} \right) e^{\frac{\sqrt{-1}}{2}\psi} \cos \frac{\theta}{2} \\ &\quad + \left(\lambda_2 \cos \frac{\kappa t}{2} + \lambda_4 \sin \frac{\kappa t}{2} \right) e^{\frac{-\sqrt{-1}}{2}\psi} \sin \frac{\theta}{2} \end{aligned}$$

and $\lambda_1, \lambda_2, \lambda_3$ and λ_4 are four arbitrary complex numbers.

Let (N, \tilde{g}) be a spacetime with negative cosmological constant Λ , and \tilde{g} satisfies the Einstein field equations

$$\widetilde{Ric} - \frac{\tilde{R}}{2}\tilde{g} + \Lambda\tilde{g} = T. \quad (2.3)$$

Suppose that T satisfies the dominant energy condition

$$T_{00} \geq \sqrt{\sum_i T_{0i}^2}, \quad T_{00} \geq |T_{\alpha\beta}|. \quad (2.4)$$

Let M be a 3-dimensional spacelike hypersurface with induced Riemannian metric g and second fundamental form h . (M, g, h) is referred as an initial data set. It is *asymptotically anti-de Sitter* of order $\tau > \frac{3}{2}$ if $g_{ij} = g(\check{e}_i, \check{e}_j) = \delta_{ij} + a_{ij}$ and $h_{ij} = h(\check{e}_i, \check{e}_j)$, and $a_{ij}, \check{\nabla}_k a_{ij}, \check{\nabla}_l \check{\nabla}_k a_{ij}, h_{ij}$ and $\check{\nabla}_k h_{ij}$ fall off as $e^{-\tau\kappa r}$ on ends, where $\check{\nabla}$ is the Levi-Civita connection with respect to the hyperbolic metric

$$\check{g} = dr^2 + \frac{\sinh^2(\kappa r)}{\kappa^2} (d\theta^2 + \sin^2 \theta d\psi^2).$$

Moreover, $T_{00}e^{\kappa\rho z}, T_{0i}e^{\kappa\rho z} \in L^1(M)$ for certain distance function ρ_z . Denote

$$\mathcal{E}_i = \check{\nabla}^j g_{ij} - \check{\nabla}_i \text{tr}_{\check{g}}(g) - \kappa(a_{1i} - g_{1i} \text{tr}_{\check{g}}(a)), \quad \mathcal{P}_{ki} = h_{ki} - g_{ki} \text{tr}_{\check{g}}(h).$$

Denote also by $U_{\alpha\beta}$ the restrictions on (1.4) of ten Killing vectors (1.5). Then we have the following total energy-momentum for asymptotically anti-de Sitter initial data sets.

$$\begin{aligned} E_0 &= \frac{\kappa}{16\pi} \lim_{r \rightarrow \infty} \int_{S_r} \mathcal{E}_1 U_{40}^{(0)} \check{\omega}, \\ c_i &= \frac{\kappa}{16\pi} \lim_{r \rightarrow \infty} \int_{S_r} \mathcal{E}_1 U_{i4}^{(0)} \check{\omega} + \frac{\kappa}{8\pi} \sum_{j=2}^3 \lim_{r \rightarrow \infty} \int_{S_r} \mathcal{P}_{j1} U_{i4}^{(j)} \check{\omega}, \\ c'_i &= \frac{\kappa}{16\pi} \lim_{r \rightarrow \infty} \int_{S_r} \mathcal{E}_1 U_{i0}^{(0)} \check{\omega} + \frac{\kappa}{8\pi} \sum_{j=2}^3 \lim_{r \rightarrow \infty} \int_{S_r} \mathcal{P}_{j1} U_{i0}^{(j)} \check{\omega}, \\ J_i &= \frac{\kappa}{8\pi} \sum_{j=2}^3 \lim_{r \rightarrow \infty} \int_{S_r} \mathcal{P}_{j1} V_i^{(j)} \check{\omega}, \end{aligned} \quad (2.5)$$

where $\check{\omega} = \check{e}^2 \wedge \check{e}^3$, $U_{\alpha\beta} = U_{\alpha\beta}^{(\gamma)} \check{e}_\gamma$, $V_i = \varepsilon_{ijk} U_{jk}$.

Now we discuss the relation between (2.5) and quantities defined in [7]. Take $\kappa = 1$ and choose $t = 0$ slice. The transformations between the hyperbolic metric $b = \frac{4}{(1-|x|^2)^2} dx^2$ used in [7] and the metric \check{g} used in our case are

$$x^1 = \tanh \frac{r}{2} \sin \theta \cos \psi, \quad x^2 = \tanh \frac{r}{2} \sin \theta \sin \psi, \quad x^3 = \tanh \frac{r}{2} \cos \theta.$$

Thus

$$\begin{aligned}\frac{\partial}{\partial x^1} &= 2 \cosh^2 \frac{r}{2} \sin \theta \cos \psi \frac{\partial}{\partial r} + \frac{\cos \theta \cos \psi}{\tanh \frac{r}{2}} \frac{\partial}{\partial \theta} - \frac{\sin \psi}{\tanh \frac{r}{2} \sin \theta} \frac{\partial}{\partial \psi}, \\ \frac{\partial}{\partial x^2} &= 2 \cosh^2 \frac{r}{2} \sin \theta \sin \psi \frac{\partial}{\partial r} + \frac{\cos \theta \sin \psi}{\tanh \frac{r}{2}} \frac{\partial}{\partial \theta} + \frac{\cos \psi}{\tanh \frac{r}{2} \sin \theta} \frac{\partial}{\partial \psi}, \\ \frac{\partial}{\partial x^3} &= 2 \cosh^2 \frac{r}{2} \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{\tanh \frac{r}{2}} \frac{\partial}{\partial \theta}.\end{aligned}$$

Now, in the polar coordinates, the quantities given in [7] are

$$\begin{aligned}V_{(0)} &= \cosh r, \quad V_{(1)} = -\sinh r \sin \theta \cos \psi, \\ V_{(2)} &= -\sinh r \sin \theta \sin \psi, \quad V_{(3)} = -\sinh r \cos \theta, \\ C_{(1)} &= \coth r \left(\cos \theta \cos \psi \frac{\partial}{\partial \theta} - \frac{\sin \psi}{\sin \theta} \frac{\partial}{\partial \psi} \right) + \sin \theta \cos \psi \frac{\partial}{\partial r}, \\ C_{(2)} &= \coth r \left(\cos \theta \sin \psi \frac{\partial}{\partial \theta} + \frac{\cos \psi}{\sin \theta} \frac{\partial}{\partial \psi} \right) + \sin \theta \sin \psi \frac{\partial}{\partial r}, \\ C_{(3)} &= -\coth r \sin \theta \frac{\partial}{\partial \theta} + \cos \theta \frac{\partial}{\partial r}, \\ \Omega_{(2)(3)} &= -\sin \psi \frac{\partial}{\partial \theta} - \frac{\cos \theta \cos \psi}{\sin \theta} \frac{\partial}{\partial \psi}, \\ \Omega_{(3)(1)} &= \cos \psi \frac{\partial}{\partial \theta} - \frac{\cos \theta \sin \psi}{\sin \theta} \frac{\partial}{\partial \psi}, \quad \Omega_{(1)(2)} = \frac{\partial}{\partial \psi}.\end{aligned}$$

Therefore we get the relation between m_ν , $c_{(i)}$ and $J_{(i)(j)}$ in [7] and our total energy-momentum

$$m_{(0)} = E_0, \quad m_{(i)} = -c_i, \quad c_{(i)} = c'_i, \quad J_{(i)(j)} = \varepsilon_{ijk} J_k.$$

By the explicit expressions of $U_{\alpha\beta}$ in appendix, we find that E_0 , J_i do not depend on t , and $\frac{dc_i}{dt} = \kappa c'_i$, $\frac{dc'_i}{dt} = -\kappa c_i$. Therefore

$$c_i = -m_{(i)} \cos \kappa t + c_{(i)} \sin \kappa t, \quad c'_i = m_{(i)} \sin \kappa t + c_{(i)} \cos \kappa t.$$

Furthermore,

$$\sum_{1 \leq i \leq 3} (c_i^2 + c'^2_i) = \sum_{1 \leq i \leq 3} (m_{(i)}^2 + c_{(i)}^2).$$

3. POSITIVE ENERGY THEOREM

In this section, we will discuss the general case of the positive energy theorem without assuming (1.1). Let (M, g, h) be an asymptotically anti-de Sitter initial data set in (N, \tilde{g}) which satisfies the dominant energy condition (2.4). Let $\tilde{\nabla}$ be the local spin connection of \tilde{g} . Define

$$\hat{\nabla}_i = \tilde{\nabla}_i + \frac{\sqrt{-1}}{2} \kappa e_i, \quad \hat{D} = \sum_{i=1}^3 e_i \hat{\nabla}_i.$$

By applying Witten's argument (e.g., [20, 19, 6, 23, 14, 7]), we can obtain the unique solution $\widehat{D}\phi = 0$ such that ϕ is asymptotic to imaginary Killing spinors Φ_0 on certain end, and to zero on other ends. Then, on this end,

$$\begin{aligned} \int_M |\widehat{\nabla}\phi|^2 + \langle \phi, \widehat{\mathcal{R}}\phi \rangle &= \lim_{r \rightarrow \infty} \int_{S_r} \langle \phi, \sum_{j \neq i} e_i \cdot e_j \cdot \widehat{\nabla}_j \phi \rangle * e^i \\ &= \frac{1}{4} \lim_{r \rightarrow \infty} \int_{S_r} (\check{\nabla}^j g_{ij} - \check{\nabla}_i \text{tr}_{\check{g}}(g)) |\Phi_0|^2 \check{\omega} \\ &\quad + \frac{1}{4} \lim_{r \rightarrow \infty} \int_{S_r} \kappa(a_{k1} - g_{k1} \text{tr}_{\check{g}}(a)) \langle \Phi_0, \sqrt{-1} \check{e}_k \cdot \Phi_0 \rangle \check{\omega} \\ &\quad - \frac{1}{2} \lim_{r \rightarrow \infty} \int_{S_r} (h_{k1} - g_{k1} \text{tr}_{\check{g}}(h)) \langle \Phi_0, \check{e}_0 \cdot \check{e}_k \cdot \Phi_0 \rangle \check{\omega}, \end{aligned}$$

where $\widehat{\mathcal{R}} = \frac{1}{2}(T_{00} + T_{0i}e_0e_i)$. Using the Clifford representation (2.1) and (2.2) for Φ_0 , we obtain the right hand side of the above formula

$$\begin{aligned} \text{RHS} &= \frac{1}{2} \lim_{r \rightarrow \infty} \int_{S_r} \mathcal{E}_1(\overline{u^+}u^+ + \overline{v^+}v^+) e^{\kappa r} \check{\omega} \\ &\quad + \lim_{r \rightarrow \infty} \int_{S_r} \mathcal{P}_{21}(\overline{u^+}v^+ + \overline{v^+}u^+) e^{\kappa r} \check{\omega} \\ &\quad + \sqrt{-1} \lim_{r \rightarrow \infty} \int_{S_r} \mathcal{P}_{31}(\overline{u^+}v^+ - \overline{v^+}u^+) e^{\kappa r} \check{\omega} \\ &= 4\pi(\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3, \bar{\lambda}_4) Q(\lambda_1, \lambda_2, \lambda_3, \lambda_4)^t, \end{aligned}$$

where

$$Q = \begin{pmatrix} E & L \\ \bar{L}^t & \hat{E} \end{pmatrix}, \quad E = \begin{pmatrix} E_0 - c_3 & c_1 - \sqrt{-1}c_2 \\ c_1 + \sqrt{-1}c_2 & E_0 + c_3 \end{pmatrix},$$

$$L = \begin{pmatrix} l_1 & l_2 \\ l_3 & -l_1 \end{pmatrix}, \quad \hat{E} = \begin{pmatrix} E_0 + c_3 & -c_1 + \sqrt{-1}c_2 \\ -c_1 - \sqrt{-1}c_2 & E_0 - c_3 \end{pmatrix},$$

$l_1 = c'_3 - \sqrt{-1}J_3$, $l_2 = -c'_1 + J_2 + \sqrt{-1}(c'_2 + J_1)$, $l_3 = -c'_1 - J_2 - \sqrt{-1}(c'_2 - J_1)$. Denote $\mathbf{c} = (c_1, c_2, c_3)$, $\mathbf{c}' = (c'_1, c'_2, c'_3)$, $\mathbf{J} = (J_1, J_2, J_3) = \vec{j}$ and

$$B = |\mathbf{c} \times \mathbf{c}'|^2 + |\mathbf{c} \times \mathbf{J}|^2 + |\mathbf{c}' \times \mathbf{J}|^2. \quad (3.1)$$

It is straightforward that

$$\begin{aligned} B &= \sum_{1 \leq i < j \leq 3} \left[(c_i c'_j - c_j c'_i)^2 + (c_i J_j - c_j J_i)^2 + (c'_i J_j - c'_j J_i)^2 \right] \\ &= |\vec{c} \times \vec{m}|^2 + |\vec{c} \times \vec{j}|^2 + |\vec{m} \times \vec{j}|^2. \end{aligned}$$

Now we can prove the following theorem.

Theorem 3.1. *Let (M, g, h) be a 3-dimensional asymptotically anti-de Sitter initial data set in spacetime (N, \tilde{g}) . Suppose (N, \tilde{g}) satisfies the dominant energy condition. Then, for each end*

$$\begin{aligned} (i) \quad E_0 &\geq \left(|\mathbf{c}|^2 + |\mathbf{c}'|^2 + |\mathbf{J}|^2 - 2|\mathbf{c}' \times \mathbf{J}| + 2C_1^{\frac{1}{2}} \right)^{\frac{1}{2}} \geq |\mathbf{c}|; \\ (ii) \quad E_0 &\geq \left(|\mathbf{c}|^2 + |\mathbf{c}'|^2 + |\mathbf{J}|^2 - 2|\mathbf{c}||\mathbf{c}' \times \mathbf{J}|^{\frac{1}{2}} + 2C_2^{\frac{1}{2}} \right)^{\frac{1}{2}} \geq \left(\frac{|\mathbf{c}'|^2 + |\mathbf{J}|^2}{2} \right)^{\frac{1}{2}}, \end{aligned} \quad (3.2)$$

where

$$\begin{aligned} C_1 &= \max \left\{ [B + |\mathbf{c}' \times \mathbf{J}|^2 - |\mathbf{c}' \times \mathbf{J}|(2|\mathbf{c}|^2 + |\mathbf{c}'|^2 + |\mathbf{J}|^2)], 0 \right\}, \\ C_2 &= \max \left\{ [B + |\mathbf{c}|^2|\mathbf{c}' \times \mathbf{J}| - |\mathbf{c}||\mathbf{c}' \times \mathbf{J}|^{\frac{1}{2}}(|\mathbf{c}|^2 + |\mathbf{c}'|^2 + |\mathbf{J}|^2 + |\mathbf{c}' \times \mathbf{J}|)], 0 \right\}. \end{aligned}$$

If $E_0 = 0$ for some end, then M has only one end, $Q = 0$, and (N, \tilde{g}) is anti-de Sitter along M .

Proof: Note that Q is non-negative. The first-order principal minors yield

$$E_0 \geq |c_3| \geq 0$$

and the second-order principal minors yield

$$E_0^2 \geq |\mathbf{c}|^2, \quad E_0^2 \geq c_3^2 + |l_1|^2, \quad (E_0 - c_3)^2 \geq |l_2|^2, \quad (E_0 + c_3)^2 \geq |l_3|^2.$$

The sum of the last three inequalities gives

$$E_0^2 \geq \frac{1}{2}(|\mathbf{c}'|^2 + |\mathbf{J}|^2) \geq |\mathbf{c}'||\mathbf{J}| \geq |\mathbf{c}' \times \mathbf{J}|.$$

Now the sum of third-order principal minors implies

$$0 \leq E_0(E_0^2 - |\mathbf{c}|^2) - E_0(|\mathbf{c}'|^2 + |\mathbf{J}|^2) + 2\varepsilon_{ijk}c_i c'_j J_k.$$

Using the Cauchy inequality, we obtain

$$\varepsilon_{ijk}c_i c'_j J_k \leq |\mathbf{c}||\mathbf{c}' \times \mathbf{J}| \leq \begin{cases} E_0|\mathbf{c}' \times \mathbf{J}|, \\ E_0|\mathbf{c}||\mathbf{c}' \times \mathbf{J}|^{\frac{1}{2}}. \end{cases}$$

Thus, when $E_0 > 0$, we have

$$E_0^2 \geq \begin{cases} |\mathbf{c}|^2 + |\mathbf{c}'|^2 + |\mathbf{J}|^2 - 2|\mathbf{c}' \times \mathbf{J}|, \\ |\mathbf{c}|^2 + |\mathbf{c}'|^2 + |\mathbf{J}|^2 - 2|\mathbf{c}||\mathbf{c}' \times \mathbf{J}|^{\frac{1}{2}}. \end{cases} \quad (3.3)$$

When $E_0 = 0$, the second-order principal minors give $\mathbf{c} = \mathbf{c}' = \mathbf{J} = 0$, so (3.3) holds trivially. Now we use the nonnegativity of the determinant of Q to prove (3.2).

$$\det Q = (E_0^2 - |\mathbf{c}|^2 - |\mathbf{c}'|^2 - |\mathbf{J}|^2)^2 + 8E_0\varepsilon_{ijk}c_i c'_j J_k - 4B$$

where B is given by (3.1). As

$$2E_0\varepsilon_{ijk}c_i c'_j J_k \leq 2E_0|\mathbf{c}||\mathbf{c}' \times \mathbf{J}| \leq \begin{cases} (E_0^2 + |\mathbf{c}|^2)|\mathbf{c}' \times \mathbf{J}|, \\ (E_0^2 + |\mathbf{c}' \times \mathbf{J}|)|\mathbf{c}||\mathbf{c}' \times \mathbf{J}|^{\frac{1}{2}}, \end{cases}$$

we obtain

$$\begin{aligned} \det Q &\leq \left(E_0^2 - |\mathbf{c}|^2 - |\mathbf{c}'|^2 - |\mathbf{J}|^2 + 2|\mathbf{c}' \times \mathbf{J}| \right)^2 \\ &\quad - 4 \left[B + |\mathbf{c}' \times \mathbf{J}|^2 - |\mathbf{c}' \times \mathbf{J}| (2|\mathbf{c}|^2 + |\mathbf{c}'|^2 + |\mathbf{J}|^2) \right], \\ \det Q &\leq \left(E_0^2 - |\mathbf{c}|^2 - |\mathbf{c}'|^2 - |\mathbf{J}|^2 + 2|\mathbf{c}| |\mathbf{c}' \times \mathbf{J}|^{\frac{1}{2}} \right)^2 \\ &\quad - 4 \left[B + |\mathbf{c}|^2 |\mathbf{c}' \times \mathbf{J}| - |\mathbf{c}| |\mathbf{c}' \times \mathbf{J}|^{\frac{1}{2}} (|\mathbf{c}|^2 + |\mathbf{c}'|^2 + |\mathbf{J}|^2 + |\mathbf{c}' \times \mathbf{J}|) \right]. \end{aligned}$$

They give (3.2).

If $E_0 = 0$, then it is straightforward that M has only one end and $Q = 0$. This implies that there exists $\{\phi_\alpha\}$ which forms a basis of the spinor bundle everywhere over M such that $\widehat{\nabla}\phi_\alpha = 0$. Standard argument gives,

$$\widetilde{R}_{ijkl} = (-\kappa^2)(\widetilde{g}_{ik}\widetilde{g}_{jl} - \widetilde{g}_{il}\widetilde{g}_{jk}), \quad \widetilde{R}_{0jkl} = 0$$

along M . The Einstein field equations yield

$$T_{00} = \widetilde{R}_{00} + \frac{1}{2}R - \Lambda = \frac{1}{2} \sum_{i,j} R_{ijij} - \Lambda = 0.$$

Then (2.4) implies $T_{\alpha\beta} = 0$ which give

$$\widetilde{R}_{0j0l} = \kappa^2 \widetilde{g}_{jl}.$$

Therefore, the Riemann curvature tensors of (N, \widetilde{g}) are

$$\widetilde{R}_{\alpha\beta\gamma\delta} = (-\kappa^2)(\widetilde{g}_{\alpha\gamma}\widetilde{g}_{\beta\delta} - \widetilde{g}_{\alpha\delta}\widetilde{g}_{\beta\gamma})$$

and N is anti-de Sitter along M .

Q.E.D.

Remark 3.1. If $E_0 > 0$, then, implicitly, \mathbf{c}' and \mathbf{J} can be chosen freely in (3.2)(i) and \mathbf{c} can be chosen freely in (3.2)(ii). In particular, we can choose $\mathbf{c}' = 0$ in (3.2)(i) or $\mathbf{c} = 0$ in (3.2)(ii), then, without assuming (1.1), (3.2)(i) at $t = \frac{\pi}{2\kappa}$ or (3.2)(ii) at $t = 0$ reduces to (1.2). Moreover, (3.2)(i) at $t = 0$ or (3.2)(ii) at $t = \frac{\pi}{2\kappa}$ gives

$$m_{(0)} \geq \sqrt{|\vec{m}|^2 + |\vec{j}|^2 + 2|\vec{m} \times \vec{j}|}. \quad (3.4)$$

This indicates that \vec{m} and \vec{c} play the same role in physics.

Finally, we remark that Theorem 3.1 holds also for the case of black holes. Suppose M has a future/past trapped surface $(\Sigma, \bar{g}, \bar{h})$ equipped with the induced metric \bar{g} and the second fundamental form \bar{h}

$$tr_{\bar{g}}(\bar{h}) \mp tr_{\bar{g}}(h|_{\Sigma}) \geq 0.$$

Let e_3 outward normal and e_A tangent to Σ . On Σ , the boundary term in the Weitzenböck formula is

$$\int_{\Sigma} \langle \phi, e_3 e_A \widehat{\nabla}_A \phi \rangle = \int_{\Sigma} \langle \phi, e_3 e_A \widetilde{\nabla}_A \phi \rangle - \int_{\Sigma} \langle \phi, \sqrt{-1} \kappa e_3 \phi \rangle.$$

Under the local boundary conditions, $\langle \phi, e_3 \phi \rangle$ is both imaginary and real, hence zero. And the first integral in the right hand is non-positive in the standard way (cf. [10]). Thus the theorem for the case of black holes follows.

4. HENNEAUX-TEITELBOIM'S ENERGY-MOMENTUM

In this section we verify that the total energy-momentum provided in our paper which arises from the boundary terms in Witten's argument is indeed the same as $J_{\alpha\beta}$ defined by Henneaux-Teitelboim [11]. Following from [11], we denote

$$\check{G}^{ijkl} = \frac{1}{2} \sqrt{\check{g}} (\check{g}^{ik} \check{g}^{jl} + \check{g}^{il} \check{g}^{jk} - 2 \check{g}^{ij} \check{g}^{kl}).$$

Henneaux and Teitelboim defined the energy-momentum as follows

$$J_{ab} = \lim_{r \rightarrow \infty} \int_{S_r} \check{G}^{ijkl} [U_{ab}^\perp \check{\nabla}_j g_{kl} - \check{\nabla}_j U_{ab}^\perp a_{kl}] dS_i + \lim_{r \rightarrow \infty} \int_{S_r} 2U_{ab}^{(k)} \pi_k^i dS_i,$$

where $\pi_k^i = \mathcal{P}_k^i$. In the orthonormal frame of (1.4),

$$J_{ab} = \lim_{r \rightarrow \infty} \int_{S_r} \check{G}^{ijkl} [U_{ab}^{(0)} \check{\nabla}_j g_{kl} - \check{\nabla}_j U_{ab}^{(0)} a_{kl}] \check{\omega} + \lim_{r \rightarrow \infty} \int_{S_r} 2U_{ab}^{(k)} \mathcal{P}_{k1} \check{\omega}.$$

As

$$\begin{aligned} & \check{G}^{ijkl} [U_{ab}^{(0)} \check{\nabla}_j g_{kl} - \check{\nabla}_j U_{ab}^{(0)} a_{kl}] \\ &= \frac{1}{2} (\delta^{1k} \delta^{jl} + \delta^{1l} \delta^{jk} - 2\delta^{1j} \delta^{kl}) [U_{ab}^{(0)} \check{\nabla}_j g_{kl} - \check{\nabla}_j U_{ab}^{(0)} a_{kl}] \\ &= \delta^{1k} \delta^{jl} [U_{ab}^{(0)} \check{\nabla}_j g_{kl} - \check{\nabla}_j U_{ab}^{(0)} a_{kl}] - \delta^{1j} \delta^{kl} [U_{ab}^{(0)} \check{\nabla}_j g_{kl} - \check{\nabla}_j U_{ab}^{(0)} a_{kl}] \\ &= (U_{ab}^{(0)} \check{\nabla}^j g_{1j} - \check{\nabla}^j U_{ab}^{(0)} a_{1j}) - (U_{ab}^{(0)} \check{\nabla}_1 tr_{\check{g}}(g) - \check{\nabla}_1 U_{ab}^{(0)} tr_{\check{g}}(a)) \\ &= (U_{ab}^{(0)} \check{\nabla}^j g_{1j} - \kappa U_{ab}^{(0)} a_{11}) - (U_{ab}^{(0)} \check{\nabla}_1 tr_{\check{g}}(g) - \kappa U_{ab}^{(0)} tr_{\check{g}}(a)) + o(e^{-2\kappa r}) \\ &= \mathcal{E}_1 U_{ab}^{(0)} + o(e^{-2\kappa r}), \end{aligned}$$

we obtain

$$E_0 = \frac{\kappa}{16\pi} J_{40}, \quad c_i = \frac{\kappa}{16\pi} J_{i4}, \quad c'_i = \frac{\kappa}{16\pi} J_{i0}, \quad J_i = \frac{\kappa}{16\pi} \varepsilon_{ijk} J_{jk}.$$

In [5], Carter obtained a family of solutions for the Einstein field equations.

$$ds^2 = \frac{\Delta_\mu (d\chi - \lambda^2 d\psi)^2 - \Delta_\lambda (d\chi + \mu^2 d\psi)^2}{\lambda^2 + \mu^2} + (\lambda^2 + \mu^2) \left(\frac{d\lambda^2}{\Delta_\lambda} + \frac{d\mu^2}{\Delta_\mu} \right),$$

where

$$\begin{aligned} \Delta_\lambda &= \frac{1}{3} \Lambda \lambda^4 + h \lambda^2 - 2m\lambda + p + e^2, \\ \Delta_\mu &= \frac{1}{3} \Lambda \mu^4 - h \mu^2 + 2q\mu + p \end{aligned}$$

where $-\infty < \chi, \lambda, \mu < \infty$, $0 \leq \psi < 2\pi$. It provides Kerr-anti-de Sitter spacetimes if $\Delta_\lambda, \Delta_\mu$ are given as follows

$$\begin{aligned}\Delta_\lambda &= (\kappa^2 \lambda^2 + 1)(\lambda^2 + a^2) - 2m\lambda, \\ \Delta_\mu &= (\kappa^2 \mu^2 - 1)(\mu^2 - a^2).\end{aligned}$$

The Kerr-anti-de Sitter solution allows $|\mu| > |\kappa|^{-1}$ and the metric has signature $(-1, 1, 1, 1)$ if $\Delta_\mu > 0$. If $m = 0$, it has constant curvature $-\kappa^2$ and reduces to anti-de Sitter spacetime.

In the region $-|\kappa|^{-1} < \mu < |\kappa|^{-1}$, $\lambda > 0$, we can take coordinate transformations

$$\lambda = \hat{r}, \quad \mu = a \cos \hat{\theta}, \quad \chi = t - a\hat{\varphi}, \quad \psi = \frac{1}{a}\hat{\varphi}$$

with $|\mu a| < 1$ and it yields Boyer-Lindquist coordinates for Kerr-anti-de Sitter spacetime

$$\begin{aligned}\tilde{g} = & - \left[1 - \frac{2m\hat{r}}{U} + \kappa^2(\hat{r}^2 + a^2 \sin^2 \hat{\theta}) \right] d\hat{t}^2 \\ & - 2a \sin^2 \hat{\theta} \left[\frac{2m\hat{r}}{U} - \kappa^2(\hat{r}^2 + a^2) \right] d\hat{t} d\hat{\varphi} \\ & + \frac{U}{\Delta_{\hat{r}}} d\hat{r}^2 + \frac{U}{\Delta_{\hat{\theta}}} d\hat{\theta}^2 + \sin^2 \hat{\theta} \frac{V}{U} d\hat{\varphi}^2,\end{aligned}$$

where

$$\begin{aligned}\Delta_{\hat{r}} &= (\hat{r}^2 + a^2)(1 + \kappa^2 \hat{r}^2) - 2m\hat{r}, \\ \Delta_{\hat{\theta}} &= 1 - \kappa^2 a^2 \cos^2 \hat{\theta}, \\ U &= \hat{r}^2 + a^2 \cos^2 \hat{\theta}, \\ V &= 2m\hat{r}a^2 \sin^2 \hat{\theta} + U(\hat{r}^2 + a^2)(1 - \kappa^2 a^2).\end{aligned}$$

It was computed in [11] that the total energy-momenta of any t -slice for Kerr-anti-de Sitter spacetime are

$$\begin{aligned}E_0 &= \frac{m}{(1 - \kappa^2 a^2)^2}, \\ c_i &= c'_i = 0, \quad i = 1, 2, 3, \\ J_1 &= J_2 = 0, \quad J_3 = \frac{m\kappa a}{(1 - \kappa^2 a^2)^2}.\end{aligned}$$

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5. APPENDIX

$$\begin{aligned}
U_{10} &= \frac{\cos(\kappa t)}{\kappa} \left[\sin \theta \cos \psi \frac{\partial}{\partial r} + \kappa \coth(\kappa r) \left(\cos \theta \cos \psi \frac{\partial}{\partial \theta} - \frac{\sin \psi}{\sin \theta} \frac{\partial}{\partial \psi} \right) \right] \\
&\quad - \frac{\sin(\kappa t)}{\kappa} \tanh(\kappa r) \sin \theta \cos \psi \frac{\partial}{\partial t}, \\
U_{20} &= \frac{\cos(\kappa t)}{\kappa} \left[\sin \theta \sin \psi \frac{\partial}{\partial r} + \kappa \coth(\kappa r) \left(\cos \theta \sin \psi \frac{\partial}{\partial \theta} + \frac{\cos \psi}{\sin \theta} \frac{\partial}{\partial \psi} \right) \right] \\
&\quad - \frac{\sin(\kappa t)}{\kappa} \tanh(\kappa r) \sin \theta \sin \psi \frac{\partial}{\partial t}, \\
U_{30} &= \frac{\cos(\kappa t)}{\kappa} \left[\cos \theta \frac{\partial}{\partial r} - \kappa \coth(\kappa r) \sin \theta \frac{\partial}{\partial \theta} \right] - \frac{\sin(\kappa t)}{\kappa} \tanh(\kappa r) \cos \theta \frac{\partial}{\partial t}, \\
U_{40} &= \frac{1}{\kappa} \frac{\partial}{\partial t}, \\
U_{14} &= \frac{\sin(\kappa t)}{\kappa} \left[\sin \theta \cos \psi \frac{\partial}{\partial r} + \kappa \coth(\kappa r) \left(\cos \theta \cos \psi \frac{\partial}{\partial \theta} - \frac{\sin \psi}{\sin \theta} \frac{\partial}{\partial \psi} \right) \right] \\
&\quad + \frac{\cos(\kappa t)}{\kappa} \tanh(\kappa r) \sin \theta \cos \psi \frac{\partial}{\partial t}, \\
U_{24} &= \frac{\sin(\kappa t)}{\kappa} \left[\sin \theta \sin \psi \frac{\partial}{\partial r} + \kappa \coth(\kappa r) \left(\cos \theta \sin \psi \frac{\partial}{\partial \theta} + \frac{\cos \psi}{\sin \theta} \frac{\partial}{\partial \psi} \right) \right] \\
&\quad + \frac{\cos(\kappa t)}{\kappa} \tanh(\kappa r) \sin \theta \sin \psi \frac{\partial}{\partial t}, \\
U_{34} &= \frac{\sin(\kappa t)}{\kappa} \left[\cos \theta \frac{\partial}{\partial r} - \kappa \coth(\kappa r) \sin \theta \frac{\partial}{\partial \theta} \right] + \frac{\cos(\kappa t)}{\kappa} \tanh(\kappa r) \cos \theta \frac{\partial}{\partial t}, \\
U_{12} &= \frac{\partial}{\partial \psi}, \\
U_{23} &= -\sin \psi \frac{\partial}{\partial \theta} - \frac{\cos \theta \cos \psi}{\sin \theta} \frac{\partial}{\partial \psi}, \\
U_{31} &= \cos \psi \frac{\partial}{\partial \theta} - \frac{\cos \theta \sin \psi}{\sin \theta} \frac{\partial}{\partial \psi}.
\end{aligned}$$

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